Secrets of the SOMAP

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Abstract:

Given the 240 solutions on the SOMAP, a program was written to generate all 480 solutions (240 plus their reflections) and produce a spreadsheet of all possible 2- and 3-piece moves between these solutions. The purpose was to search for unused connections and possible swapping with reflected solutions that could enhance the appearance of the SOMAP. With the large number of
 moves with three combinations shown, it was thought there must be more combinations of single
 moves shown near the lower-left corner of the SOMAP. Knowing solutions B2f and B4f have connections to reflected solutions, the thought was to investigate similar connections elsewhere on the SOMAP.

The main finding was that the only 2-piece connections between the real-world and reflected SOMAPS relate to the B2f/B4f reflected pair of solutions. This led to the possibility of an animated scene involving transitions of these solutions in front of a mirror that can be switched on and off. This could be developed as a science fair project to get people interested in the SOMA Cube.

Another interesting finding is a third 4-piece move between solution R7c and R7d, the Diamond.

Introduction:

This discussion assumes you're familiar with the structure and terminology associated with the SOMAP. You should at least have read SOMAP For Dummies, and SOMAP Comments[1] by Merv Eberhardt, available on Thorleif's SOMA page. Note that much of the methodology used to generate the SOMAP and described below is speculation, based on analysis of the SOMAP itself. I apologize if anything is in error and would like to hear from you if you know it occurred differently.

According to professor Donald E. Knuth of Stanford University, Richard K. Guy found 234 solutions to SOMA Cube and started the SOMAP back in 1960[2]. Twelve years later, Vol. 2, #2 of the SOMA Addict[3] presented a portion of the SOMAP and an interview with John H. Conway. It says the SOMAP is "the unique conception and development of Dr. J. H. Conway." This doesn't mention Richard Guy's involvement, although his article in the journal NABLA, when he was in Malaya, is mentioned. The SOMA Addict article goes on to quote Conway saying "We (JHC and his Cambridge colleague M. J. T. "Mike" Guy) enumerated the 240 (basic) SOMA solutions *by hand* . . . one wet afternoon . . . ". Note that Michael J. T. Guy is the son of Richard Guy.

Regardless of the exact date of development, the SOMAP is a beautiful work of mathematics and art. The 240 cube solutions are acted on by the set of all 2- and 3-piece operations where the pieces can be rearranged to fill the same configuration of small cubies. Each operation, such as
 acts on a cube solution containing it, and through its action connects to another solution. These are partial operations since each one only works on some of the cube solutions. These operations are the links shown on the SOMAP. Richard Guy, John Conway and Michael Guy probably used these operations (we'll call them moves) to generate a table of solutions and associated moves, and then used the table to create the SOMAP.

Now that Merv Eberhardt has navigated the SOMAP and published all 240 solutions[1], I thought it would be interesting to search the map in detail for possible links to the 240 reflected solutions. I wrote a program to reverse-engineer the process of making the SOMAP and generate a spreadsheet of all possible 2- and 3-piece moves amongst the 240 basic solutions and their 240 reflected solutions. The spreadsheet pointed out several interesting conclusions as noted in the Results section.

Discussion:

There are actually 480 solutions to the 3x3x3 SOMA cube, but they occur as 240 pairs of reflected solutions. There are many ways to choose which solution of each pair is in regular space and which is in reflected space. With 240 binary choices there are 2^240 ways to separate the solutions to 240 in regular space and 240 in reflected space. If a grain of sand represents one possible way to define a regular set of 240 solutions, the pile of 2^240 grains would occupy a sphere of sand (if it could physically exist) about 38,000 light years in diameter, or about 1/3 the diameter of the Milky Way!

There are many ways to define the set of 240. One, used by Christoph Peter-Orth in his article "All solutions to the SOMA cube puzzle", notes that piece A always touches the left or right side of the cube, with all solutions normalized with the G piece fixed in the front of the bottom layer. The simple rule "Piece A touches the left side" reduces the 2^240 possibilities to one. This is an elegant way to define the set of regular solutions, but it only works well for computer-generated solutions. For manual generation, a method providing easy checking for duplicate and reflected solutions is needed. It appears the SOMAP solutions were generated starting with one solution and then moving to other solutions by finding configurations of two pieces that fill the same space when the pieces are swapped. All 2-piece and some necessary 3-piece moves were used. Even one 4-piece move was needed. The example below shows the three configurations of a
 move, which occurs in numerous places in the lower center of the SOMAP. If a reflected solution was found, it could be rejected and they'd move on.

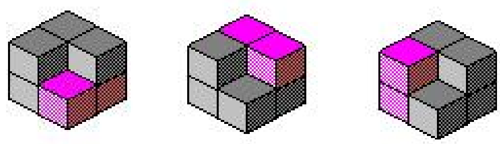

Figure 1: Three orientations of a
 move

Using the SOMATYPE designation with Deficiency, Centality, and Dexterity followed by an a,b,c, … suffix, only the existing list of the same SOMATYPE needs to be checked for reflected solutions. Either a SOMATYPE or it's reflected designation can be used in a sequence, so you see on the SOMAP that the a,b,c, … suffix in each SOMATYPE chain varies between A2/A5, B0/B6, R6/U1, BR2/BU4, etc. There are at least 17 pairs of reflected SOMATYPES in chains on the SOMAP. Either type of the pair is fine to have in regular space as long as it isn't a reflection of an existing solution.

When developing the SOMAP all 2-piece moves were used, but this resulted in 40 "islands" of solutions not connected to each other. On the SOMAP shown in the SOMA Addict there are letters written in pencil to show these islands. They range from 119 solutions for A (including the one reflected solution), down to two each for islands R-Y. Z was used to represent all 15 isolated, single-solution islands. 3-piece moves were then found to connect the islands to each other.

The SOMAP shows the set of 240 solutions generated in this way, plus one reflected solution. There is also a reflected SOMAP containing the reflected solutions. To see this, simply hold the SOMAP up to a mirror, and there it is. When reading SOMATYPES and move designations, it's necessary to follow the rules of reflection (R and U swap, and dexterity becomes its complement). In this way both SOMAPS can be examined. When thinking about generating solutions and moving about the SOMAP it helps to think of doing this in front of a mirror. Your reflected self is then generating the reflected SOMAP in the mirror and similarly separating the solutions into regular and reflected space.

To generate all possible 2- and 3-piece moves between the 480 solutions, a computer program was written to read the 240 solutions and generate the 240 reflected solutions. It then compares each numbered solution against all higher-numbered solutions. Each comparison rotates the first cube through the 24 possible rotations and compares this to the second cube. The 27 small cubies in the 3x3x3 cube are compared to see which SOMA pieces are in different positions. If only two or three are different, the pieces are noted as a move between these two solutions. This involves tens of millions of comparisons, but the program runs in under two seconds. Note that solution B2f on the SOMAP (one of the reflected pair shown) was moved to reflected space so two sets of 240 solutions would be generated. The output is a comma-delimited list that is then imported into a spreadsheet for examination. The spreadsheet has the following format:

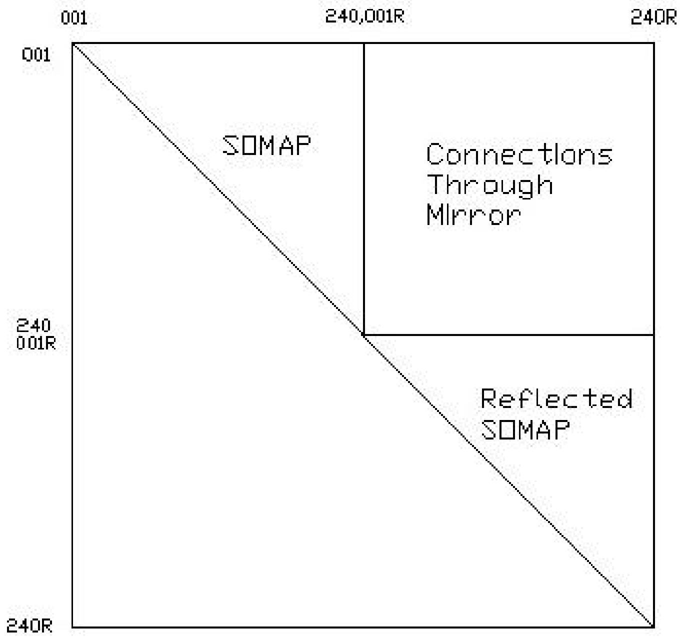

Fig. 2: The three sections of the spreadsheet

Note the terminology for the reflected solutions varies from the regular SOMATYPE by using the designation of the solution in regular space followed by "R". This makes it easy to tell if a solution is in normal space or reflected space. When B2f was moved to reflected space its designation changed to B4fR. The spreadsheet is available at:

https://www.fam-bundgaard.dk/SOMA/NEWS/N181006%20SOMAP%20Moves.xlsx
"Table of all possible 2- and 3-piece moves."

A spreadsheet was also developed showing only possible 2-piece moves on the SOMAP. It was only used to show the program agrees with the 308 2-piece moves shown on the SOMAP, and to gather some statistics. Also, a list was created of all 4-piece of moves connecting to the single R7d (the Diamond) solution.

Results and Observations:

The first area examined was the upper-right portion of the spreadsheet, showing all moves between the SOMAP and reflected SOMAP through the mirror. The surprising result is that the only 2-piece moves are connected to B4f and B2f (B4fR). This is not just one example of a pair of solutions producing 2-piece moves through the mirror; it's the ONLY one. Regarding 3-piece moves there are hundreds through the mirror, as well as hundreds unused (and unnecessary) on the SOMAP.

As written, the SOMAP has six 2-piece moves connecting through the mirror that are not shown. There are two from B2f to the reflections of O1f and B4e (connected to B4f on the SOMAP), and four from B4f to the reflections of solutions connected to B2f on the SOMAP. These can all be seen with a correct reading of the reflected SOMAP.

Similarly, the reflected SOMAP has six connections through the mirror back to regular space.

It is possible to remove all 2-piece moves through the mirror, using a 3-D manipulation of B2f and B4f. In a thought experiment, lift B2f and B4f from the paper and change their six connecting lines (to other solutions) into strings between them and the paper. Place the two solutions on top of each other and add an additional label to each string indicating which solution is connected to it (B2f or B4f). Now hold the SOMAP up to a mirror and place the B2f/B4f solutions on the face of the mirror. Your reflected self is doing the same, and the result is the SOMAP with six strings connecting to B2f/B4f at the mirror surface, as well as the reflection of this showing reflected strings connecting to the reflected SOMAP. All labels on the reflected strings need to be read in the usual reflecting manner, but there are now no 2-piece connections through the mirror. With the special status of B2f/B4f, they are discussed more in the Future Possibilities section.

The next interesting point relates to the Diamond solution R7d. The spreadsheet confirms it does not have any 2- or 3-piece connections. The 4-piece list shows 14 solutions other than R7c that connect to it in real space with a single 4-piece move. It's known there are two 4-piece moves from R7c to R7d, as stated in the clues "When you find the diamond, bury it again – so it's secret is left!" (from the SOMA Addict), and "The diamond's gory secrets are seven seas away!" (from Winning Ways for Your Mathematical Plays, Volume 4)[4]. The list shows there is even a third 4-piece move from R7c to R7d. I'll dispense with hints and just say it's a <bugo> move. With 4-piece moves I'm not surprised there are 17 from various solutions, but I'm amazed there are three 4-piece moves between these two solutions. It's interesting to note that no two solutions of the 480 are connected by more than one 2- or 3-piece move.

Each of the 4-piece moves between R7c and the Diamond is special in another way. Most 4-piece moves involve reorienting each of the 4 pieces, but these moves involve pairs of pieces that maintain their orientation within each pair. Merv Eberhardt noted that each of these moves involves two pairs of SOMA pieces that are effectively glued together. By gluing the BU, GO, and RY pairs together we are now dealing with 4 polycubes forming the 3x3x3 cube. The BU piece has 7 cubies, GO and RY have 8 cubies each, and the remaining A piece has 4. The three moves between the solutions are <BU + RY>, <GO + RY>, and <BU + GO>.

Assembling three cubes with the R7c configuration using these glued pairs demonstrates the symmetries involved with these moves. Place a white dot on one of the three edges occupied by the A piece (same edge on each of the identical R7c cubes). Now perform a different one of the three moves on each cube. Finally, rotate the transformed cubes by 0, 120 or 240 degrees about the diagonal axis from the center through the vertex occupied by the A piece, so the SOMA G piece is in the normalized position at the bottom front of the cube. All three cubes are now in the R7d (Diamond) configuration, but each of the white dots is now on one of the three different edges occupied by the A piece. The dots highlight each of the three rotational symmetries of the A piece.

As noted in the Abstract, a main reason for producing the spreadsheet was to search for 2-piece moves connecting through the mirror. After finding essentially none, the question remains, what happened to the other
 moves when only a single one is shown on the SOMAP? It turns out there are several forms of
 moves (and others), and the single connections represent a form with only two possible configurations. Not wanting to give up completely, the focus shifted to 3-piece moves.

There are 588 possible 3-piece moves connecting through the mirror. Remembering the 40 islands of solutions, they are all unique via 2-piece moves within each island. However, entire islands could be relocated and swapped with their twin in reflected space. Doing this eliminates 3-piece connections shown on the existing SOMAP, but new connections are now available to what was previously the mirrored island. Imagine you and your reflected self reaching through the mirror, grabbing an island (and its reflection) and pulling them back through the mirror so they've switched from one SOMAP to the other. A few of these swaps were tried, connecting to completely different areas of the SOMAP, but none seemed to provide a better-looking map.

The spreadsheet shows 651 possible 3-piece moves on the SOMAP, but far less than these were actually used. It appears all 3-piece moves used are connecting islands, other than one exception between A0f and A6a near the right edge of island A. The large number of unused moves is not a surprise since they only used a few more than enough to link the islands together, but the additional moves provide an opportunity to highlight other connection patterns. The SOMAP in Winning Ways already has three additional 3-piece <ury> moves (actually different piece colors on that map) near the upper-right corner between A6d and A6g, O3f and O3h, and A3f and A3h. These create cubes of solutions in this area, similar to the ones at their immediate right.

Other 3-piece moves that potentially enhance the appearance of areas are:

- Near the middle-bottom, a <bra> move between BU6a and RU6d makes a nice pentagon of RU6d, RU6c, RU5d, BU4d, and BU6a.
- If the SOMAP was converted to three dimensions to highlight existing triangles, hexagons, prisms, etc. the right side of island A could be raised to make the two long diagonal lines in the center of the map horizontal. The two long rows of solutions could then be wrapped around in three dimensions and connected with <boy> moves between U2a and U0e, and RU2a and RU0e. This band of solutions forms a nice 9-sided (Nonagon) figure.
- To the right of middle-bottom, a <bgy> move between O1k and O3d produces a by/bg/gy/bgy square of solutions.
- To the left of middle-bottom, a <bur> move between U3d and RU3e completes a bu/ru/br/bur square.

There are many more observations, statistics and interesting evidence of the order in which solutions were determined, but those can wait for a future article.

Future Possibilities:

Getting back to the B2f/B4f reflected pair with 2-piece connections between real and reflected space, an interesting animation is possible. The special status of these solutions needs some demonstration of the mirrored properties. I don't have the software to do this, but perhaps someone in the animated film industry will be interested. This is based on the B4e/B4f relationship first noted by Conway in the SOMA Addict article:

The scene starts with a person sitting at a table with a B4e cube and B4f cube on it, and a mirror at the back of the table. The mirror shows the person, B2e and B2f in the reflection. These cubes all remain fixed, and there's a third cube (and reflection) sitting next to B4e, also with B4e configuration.

1. The person picks up the third cube and does a B4e-<bg>-B4f transition in real space, mirrored as B2e-<bg>-B2f by his reflection. He then rotates it to the normalized position and places it next to B4f, which it now matches.

2. Next, he picks it up again and does a B4f-<oy>-B2e transition and tries to move it diagonally through the mirror toward B2e. His reflection is doing the similar transition (B2f-<oy>-B4e). It bumps up against the mirror (reflection trying the same thing) so he puts the cube down close to the mirror. Reaching up, he flips a switch next to the mirror. This is a magic mirror that switches off, becoming an empty frame. His reflection disappears and the background through the frame changes. The cubes on the table (now 6 individual cubes) do not change at all.
3. He now picks up his moveable cube, moves it diagonally through the frame and places it next to B2e. Then he picks up what was the reflected cube near the mirror and moves it diagonally into his space, placing it next to B4e.
4. Finally, he reaches up and flips the switch to turn the mirror back on. The reflections return and everything looks like the beginning of the scene.

This could be a science fair project to get people interested in the SOMA Cube and SOMAP. The animation could run as a continuous loop with several real cubes nearby so people could do the transitions themselves.

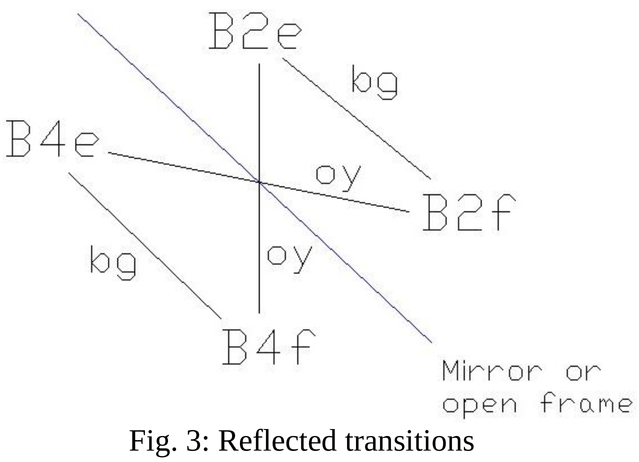

Fig. 3: Reflected transitions

Another possibility for future work is more investigation of island swaps and additional 3-piece moves to add structure to the SOMAP. Generation of a 3-D SOMAP using software designed to make stick-and-ball constructions of large proteins and other molecules is possible as well.

References:

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