

relation function; and (3) finally calculate the inverse Fourier transform of the modulus squared of the Fourier transform, perhaps using an inverse fast Fourier transform algorithm. The result is the autocorrelation function of the noise, but this method is much faster than the direct calculation of the autocorrelation function described previously. Students will be impressed by the speed of this important numerical technique.

## V. CONCLUSION

The experiment described here creates an opportunity for students to employ important concepts from electrical circuits and Fourier analysis, while gaining experience with digital electronics and modern computer-aided data acquisition. Although only one experiment is described, there are many possibilities as soon as a noise generator is connected to a data acquisition computer. Besides the noise experiments suggested in Sec. I, simple Fourier analysis of the noise, perhaps employing the FFT algorithm, would be a very illustrative exercise. Once this capacity is present, the introduction of filters would allow for further exploration of basic electric circuitry. At this point the computer would be functioning as a spectrum analyzer, which may not make sense if

a spectrum analyzer is available. Still, one of the advantages in all of these experiments is that the student does all of the programming without devoting a large amount of valuable laboratory time to programming.

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## Heisenberg's lattice world: The 1930 theory sketch

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About 1930, physicists were increasingly frustrated about the infinities of quantum electrodynamics and the strange behavior of what were believed to be nuclear electrons. As a way out of the problems Heisenberg suggested that space be subdivided in cells of finite size, and indicated in a letter to Bohr the essence of his theory. In Heisenberg's lattice world, the electron could metamorphose into a proton, and the atomic nucleus consisted of protons and heavy "photons." We analyze Heisenberg's fascinating (but unpublished) theory in its historical context, and suggest a detailed reconstruction of the lattice world idea contained in the letter to Bohr. © 1995 American Association of Physics Teachers.

## I. INTRODUCTION

In a paper appearing in the fall of 1930, dealing with the infinite self-energy of the electron, Werner Heisenberg included the following remark<sup>1</sup>

[It would seem] plausible to introduce the radius  $r_0$  [of the electron] in such a way that space is divided into cells of finite magnitude  $r_0^3$ , and the previous differential equations are replaced with difference equations. In such a lattice world the self-energy will, at any rate, be finite. However, although such a lattice world possesses remarkable properties, one must also observe that it leads to deviations from the present theory which do not seem plausible from

the point of view of experiment. In particular, the assumption that a minimal length exists is not relativistically invariant, and one can see no way to bring the demand for relativistic invariance into conformity with the introduction of a fundamental length.

Most readers of the issue of *Zeitschrift für Physik* probably found this comment rather cryptic, for Heisenberg did not give the slightest hint of either its context or how he had derived the results of a cellular space. What he had in mind was, in fact, a theory sketch which he had worked out earlier in the year and communicated to Bohr in a private letter. In the letter, Heisenberg suggested that the world is structured

as a lattice—that it is a *Gitterwelt*. He believed that he could in this way solve some of the frustrating problems of quantum electrodynamics and also produce a theory based on only one kind of elementary particle. In this sense, Heisenberg's theory aspired to the same high goals as the hole theory suggested by Paul Dirac at the same time. (Although Heisenberg's work, given its unfinished and tentative formulation, should properly be referred to as a theory sketch, or an idea, for linguistic reasons we will refer to it also as a "theory.")

Furthermore, Heisenberg's lattice world also promised to cast new light on the structure of the atomic nucleus and to get rid of the electrons that were supposed to reside in the nucleus but which were unwelcome because of their strange behavior. Altogether, the theory would have heralded a revolution in physics, had it worked. It did not.

The radical idea was met with skepticism from Bohr and others, and after one or two months Heisenberg decided to bury it. For this reason the theory remained unpublished, and it has only survived in the form of the mentioned letter to Bohr.<sup>2</sup> In spite of its brief life the theory was rather widely discussed among insiders in the physics community, and some of its basic ideas were later revived by Heisenberg. The unpublished sketch of 1930 was an ingenious and original attempt to revolutionize microphysics, with elements that are highly interesting both from a historical and a physical point of view. Among these are the impact of ideas from solid state physics, undoubtedly the first case of transfer of methods from solid state to elementary particle physics.

The present paper locates Heisenberg's theory of 1930 in its proper historical context, including not only quantum electrodynamics and solid state theory but also nuclear physics and the more speculative theories of a discrete space-time structure that existed in the late 1920s. The core of the paper consists of a translation of Heisenberg's letter and a detailed reconstruction of its meaning in terms of physical theory known at the time. By means of this reconstruction we believe we have explained, in a technical sense, Heisenberg's theory of a lattice world. As all reconstruction, our attempt is in part based on rationalization and some guesswork, which in this case is unavoidable because of the absence of supplementary source material: Together with most other sources, Heisenberg's calculations and notes concerning the lattice world disappeared when his institute in Leipzig was raided during the war.

## II. QUANTUM THEORY AND DISCRETE SPACE TIME

In 1929, Heisenberg and Wolfgang Pauli had developed an elaborate, relativistically invariant theory of quantum electrodynamics that was to become the foundation of the field for the next decade.<sup>3</sup> However, it almost immediately turned out that in this theory the calculation of some quantities also gave an infinite result. Later, at the end of 1929, Heisenberg, Pauli, Oppenheimer, and a few others realized that the self-energy of the point-like electron remained infinite (as in classical electrodynamics) and that the new quantum electrodynamics was therefore in trouble. The question of infinities in quantum electrodynamics deeply concerned Heisenberg who was constantly looking for methods to improve the theory.

One possible way to avoid (some of) the infinities was to postulate a smallest length and also, perhaps, a smallest time interval. Closely related to this idea were various attempts of

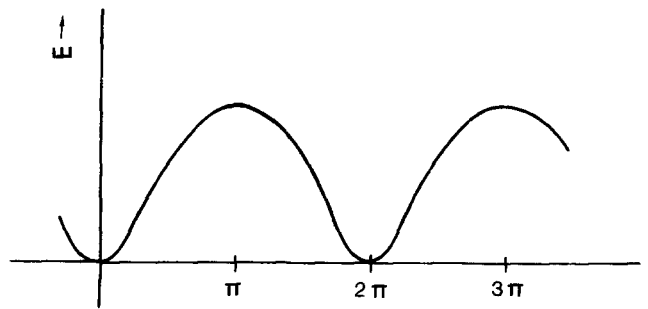


Fig. 1. Heisenberg's sketch of the variation of the energy of an electron with the "quantum number"  $ka$ , as included in his letter to Bohr of March 10, 1930.

the late 1920s to introduce a discrete space time, i.e., to conceive space and time as consisting of smallest finite units. Such reasoning was followed by several physicists who, on a more or less speculative basis, suggested the existence of time atoms or "chronons." For example, in a highly speculative paper of 1929, Gottfried Beck, a Swiss amateur physicist, suggested that all particles were intermittently moving with the speed of light, traversing one space atom per time atom.<sup>4</sup> A related idea, first suggested by Henry Flint and Arthur Ruark in 1928, was to conceive of the uncertainty in the position of a particle as having an absolute minimum value, independent of the uncertainty of the momentum.<sup>5</sup> For electrons, the value was taken to be  $h/mc$ , and, in the case of protons,  $h/Mc$ , with  $m$  and  $M$  denoting the respective masses. About 1930, the idea of absolute uncertainties was widely discussed, often in the form of  $h/Mc$  being a kind of minimum length below which quantum mechanics would cease to apply. In one version or another, the hypothesis was accepted by leading quantum physicists such as Bohr, Pauli, Schrödinger, Heisenberg, de Broglie, and Jordan.

Neither Beck, Ruark, nor Flint were concerned with quantum electrodynamics, but in the summer of 1930 the two young Soviet physicists Dmitri Iwanenko and Victor Ambarzumian transferred the idea into quantum theory. They conceived space as a three-dimensional, cubic lattice in which case the ordinary differential equations of quantum mechanics had to be replaced by difference equations. However, the plan was not developed very far. Considering the Heisenberg–Pauli expression for the Coulomb self-energy, they simply replaced the Green function of the Laplace operator by that of the corresponding finite difference operator and in this way obtained a finite value of the electron's self-energy. They also inferred the existence of a minimum time interval  $\Delta t = \Delta x/c \sim e^2/mc^3$ . The inverse of this time atom would be a maximum frequency, which Iwanenko and Ambarzumian assumed would correspond to the hypothetical proton–electron annihilation process  $p^+ + e^- \rightarrow 2h\nu$ . Therefore

$$(M+m)c^2/2h \sim mc^3/e^2$$

so that  $m/M$  and the reduced fine structure constant  $e^2/hc$  are of the same orders of magnitude, in agreement with experience.

In a note added in proof, the two Soviet physicists wrote: "Heisenberg has attempted an analogous quantization [of space]. He has succeeded in integrating the difference wave equation of the free electron. In this way he obtained the

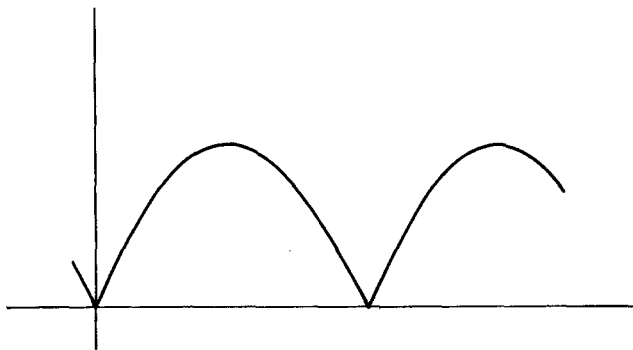


Fig. 2. Heisenberg's drawing of the variation of the energy of a neutral, massless particle ("photon") with the quantity  $ka$ . The axes are the same as in Fig. 1.

most remarkable result of maximum eigenvalues."<sup>6</sup> But, as mentioned, although Heisenberg communicated his work to Ambarzumian and Iwanenko, among others, it never reached the pages of *Zeitschrift für Physik* or any other journal.

### III. A COMPLETELY MAD IDEA?

Heisenberg's idea of considering space cells of finite size was clearly introduced in order to solve fundamental problems in quantum electrodynamics. It may have been inspired by some of the earlier speculations about discrete space time, such as Beck's, and possibly also by Dirac's unitary theory of protons as the absence of negative energy electrons.<sup>7</sup> The terminology and notation used by Heisenberg in his letter may suggest that he considered space as a discrete manifold with the wave functions defined only on the points of a spatial lattice. However, as far as we have understood the theory sketch, this is not the correct interpretation. In effect, Heisenberg considered continuous position coordinates and wave functions defined on a continuous manifold: it is only the derivatives in space coordinates that are replaced by finite differences by the introduction of an elementary length. Therefore, Heisenberg's use of the term "lattice world" is unfortunate since it does not reflect the true meaning of the theory.

Heisenberg first reported about his idea in a letter to Bohr of February 26, 1930: "I now believe that in the electrodynamics of Pauli and myself the self-energy of the particles, and also the Dirac transitions, spoil everything. I have lately tried, as earlier with the phase space, to divide the real space in discrete cells of magnitude  $(h/Mc)^3$  ( $M$ =mass of proton) and in this way to obtain a reasonable—but not, of course, a quantitatively useful—theory in the sense of correspondence considerations."<sup>8</sup> His idea was to utilize "the freedom which is given by the uncertainty of  $h/Mc$  for all lengths"—a reference to the idea of absolute uncertainty. How to utilize the freedom was reported two weeks later, in another letter to Bohr.<sup>9</sup> The main part of this letter, the key document for understanding Heisenberg's early lattice world, is translated below.

As you know, the self-energy of the electron causes the most terrible things in Pauli's and my electrodynamics; in particular one also gets here contributions to the self-energy which have to do with the Dirac transitions, and therefore a complete confusion. Since now, on the other hand, a length can presumably not be determined more precisely than  $h/Mc$  ( $M$  is the proton mass), then there

seems to me to be a freedom here which perhaps can be used to remove the difficulty with the self-energy. The most primitive method to account for this uncertainty  $h/Mc$  in a correspondence-like way, so to speak, seems to me to be a division of space into cells of the magnitude  $(h/Mc)^3$ . So, I have asked myself what such a lattice world looks like. For this purpose I initially calculate one-dimensional problems. Instead of the Gordon-Klein equation  $\square\phi + m^2c^2\phi = 0$ , something like

$$-(E/c)^2 + (h/2\pi ia)^2[u_{n+1} - 2u_n + u_{n-1}] + m^2c^2u_n = 0 \quad [1]$$

would then appear, with  $a$  being the cell radius. (Here, the time is initially treated just as usually.) If the energy is pictured as a function of the quantum number, one obtains the following curve (Fig. 1). In the neighborhood of the minima ( $E/c \sim mc$ ), the electron thus behaves quite normally. On the other hand, near the maxima (in continuous space there are no maxima) it behaves like a proton. This is seen as follows: First, the curvature of the curve corresponds to the mass  $h/2\pi ac \sim M$ , which follows from Eq. [1] (i.e., identical with  $M$  for an appropriate choice of  $a$ ); next, the curvature is negative, i.e., the electron responds to external forces like a positive charge (cp. with Peierls' work on the anomalous Hall effect). Finally, it also acts field-producing like a proton; with a reasonable choice of the Lagrangian function,  $-e(u_n^*u_{n+1} + u_{n+1}^*u_n)$  enters as a charge density [and] in the neighborhood of the maximum [we have] just that  $u_{n+1} \sim -u_n$ . I have calculated these properties more exactly. I further believe that the mass term of the electron  $mc^2$  in Eq. [1] can be dispensed with. The self-energy provides the electron with a mass, and, to be more exact, the order of magnitude of this energy is  $e^2/a$ ; i.e.,  $e^2Mc^2/hc$ . In such a lattice world the mass term can thus, most probably, be cancelled, and the electron's mass will be given by theory; in that way,  $m/M \sim e^2/hc$ . Of course, it would make no sense to calculate the numerical coefficient.

Among the further consequences of the lattice world is the following: The curve of the energy as a function of the quantum number looks like (Fig. 2) also for light quanta. That is, the group velocity disappears in the vicinity of the maximum  $\lambda \sim h/Mc$ . The velocity of light must thus be a function of the frequency, the deviations from  $c$  being of the order of  $(h\nu/Mc^2)^2$ . Also the charge of the electron would be a function of the velocity, where the deviations also in this case would be of the order of  $(h\nu/Mc^2)^2$ .

If the lattice world is extended to three dimensions, it appears that it becomes difficult to make space really isotropic. Also there is no decent Lorentz invariance. Furthermore, energy and momentum conservation are not valid in such a lattice world (i.e., only modulo  $2\pi$  with respect to the quantum number; cp. with Peierls' lattice works). Neither is there any charge conservation. That is, all the laws are approximately valid in the ordinary atomic physics, but they break down in nuclear physics. Another interesting result is this: The atomic nuclei consist only of protons and (slow) light quanta of mass  $M$ , not of electrons. For in order to build up wave packets of nuclear dimensions, only waves close to the maximum of the  $E$  curves can be used.

I do not know if you find this radical attempt completely mad. But I have the feeling that nuclear physics is not to be had much more cheaply. Of course I do not take

this particular lattice model very seriously: It is only meant as a help in seeing what the qualitative effects are of introducing a universal length. As I wrote, I believe these qualitative results to be open for discussion, and I would not be surprised if something of this kind was found experimentally.

This was as much as Heisenberg wrote to Bohr. In spite of the length of the letter, Heisenberg's reasoning is not clear at all, and certainly not immediately. In particular, how did he obtain the results reported with such confidence? How reliable are they (and were they)? In order to settle these questions—and to settle our own curiosity—we offer an interpretation of Heisenberg's approach.

## IV. RECONSTRUCTION OF HEISENBERG'S APPROACH

### A. Energy spectrum

In order to derive the results reported qualitatively by Heisenberg, we start with the Klein–Gordon equation for a free electron, as written by Heisenberg, namely,

$$\square \phi + m^2 c^2 \phi = 0, \quad (1)$$

where  $m$  denotes the rest mass. Heisenberg knew of course that this equation does not describe real electrons, but in order to avoid complications with spin at this first stage of his research program he chose to consider spinless electrons instead of using the Dirac equation. In the one-dimensional case considered by Heisenberg the d'Alembert operator  $\square$  is  $\hbar^2(c^{-2}\partial^2/\partial t^2 - \partial^2/\partial x^2)$  which gives

$$\hbar^2(\partial^2 \phi / \partial x^2 - c^{-2} \partial^2 \phi / \partial t^2) = m^2 c^2 \phi. \quad (2)$$

Since time is treated “as usually,” we write

$$\phi(x, t) = u(x)f(t) = u(x)\exp(-iEt/\hbar)$$

and obtain

$$\hbar^2 \partial^2 u / \partial x^2 - (E/c)^2 u = m^2 c^2 u. \quad (3)$$

Now the question of discreteness enters. We first remark that Heisenberg only considers discreteness in space and that he treats time as usually, i.e., as a continuous parameter. His use of the word “initially” (*zunächst*) may indicate that he was considering, in a further development of the theory, to also introduce discreteness in the time variable, but this is not done in the letter. However, as we shall see in Sec. IV D, there are reasons to believe that Heisenberg did also consider time operators to be discrete.

With regard to the space function  $u(x)$  it is not very clear whether Heisenberg defines it only on the points of a lattice, i.e., only for  $x=0, \pm a, \pm 2a$ , etc., or if he considers  $u(x)$  for all continuous values of  $x$  and thus introduces the elementary length only by means of a discrete operator replacing the differential operator. We have adopted the second possibility which makes better sense in a solid state theoretical context and which also agrees better with Heisenberg's later theory of a smallest length.

In that case, we define  $x$  as a continuous parameter,  $x = \xi + na$  with  $n=0, \pm 1, \pm 2, \dots$  and  $0 \leq \xi \leq a$ . The variable  $u(x)$  is then replaced by  $u_n(x) = u(\xi + na)$ . In order to explore the space-lattice world, Heisenberg makes use of a second-order difference quotient as a substitute for the differential quotient  $\partial^2/\partial x^2$ , namely

$$\Delta^2 u(\xi + na) = \{u[\xi + (n+1)a] - 2u(\xi + na)$$

$$+ u[\xi + (n-1)a]\}/a^2. \quad (4)$$

With this substitution, Eq. (3) just becomes Eq. [1] in Heisenberg's letter. We now have to solve the equation, for which purpose we write

$$u_n(x) = \exp[ik(\xi + na)], \quad (5)$$

where  $k$  is the wave number,  $k = 2\pi/\lambda = p/\hbar$ . The result of the substitution is that Eq. [1] yields

$$-(\hbar/a)^2[e^{ika} - 2 + e^{-ika}]u_n + m^2 c^2 u_n = (E/c)^2 u_n. \quad (6)$$

Euler's formula then gives the energy as a function of  $k$  and the length  $a$ :

$$(E/c)^2 = m^2 c^2 + 2(\hbar/a)^2(1 - \cos ka). \quad (7)$$

As in the case of the ordinary Klein–Gordon equation, the energy of the free electron can attain both positive and negative values. Heisenberg considers only the positive root

$$E = [m^2 c^4 + 2(\hbar c/a)^2(1 - \cos ka)]^{1/2}. \quad (8)$$

For a free particle described by the ordinary Schrödinger equation in one dimension the plane waves with the same absolute value of the momentum, but moving in opposite directions, have the same energy. From Eq. (8) it follows that the energy is an even function of the wave number, so in the lattice world we also have the same energy for  $\pm k$ .

A characteristic feature of the present case is that the energy eigenvalues are further degenerate. In fact, since they are given by a periodic function of the quantity  $ka$  we have

$$E(k) = E(k + n2\pi/a), \quad (9)$$

where  $n=0, \pm 1, \pm 2$ , etc. All the values attained by the energy are those corresponding to the first Brillouin zone  $-\pi/a \leq k \leq \pi/a$ . As a consequence of the degeneracy a solution corresponding to the energy eigenvalue  $E(k)$  can in general be written as a linear superposition of waves of the kind

$$\exp(ikx) \sum_n c_n \exp(in2\pi x/a),$$

where the coefficients  $c_n$  are undetermined. In general,

$$u(x) = \exp(ikx)w(x),$$

where  $w(x) = w(x+a)$  is any periodic function with period  $a$ . This is just the form of the Bloch solution for an electron in a periodic potential, introduced by Felix Bloch in his seminal thesis work of 1928 done with Heisenberg as his supervisor.<sup>10</sup> The product  $\hbar k$  of Planck's constant and what Heisenberg calls the “quantum number” (the wave number) thus corresponds to the quasi (or crystal, or lattice) momentum known from solid state physics. The analogy is not surprising since the Hamiltonian for an electron in a periodic potential and the finite difference operator introduced by Heisenberg have the same symmetry properties, i.e., both are invariant under a finite translation. However, in the solid state case the periodic factor  $w(x)$  is known since the energy eigenstates are well defined. In the present case the form of  $w(x)$  cannot be further specified.

The values of  $E$ , as found above, oscillate between

$$E = mc^2 \quad (\text{for } k=0 \pm n2\pi/a) \quad (10)$$

and

$$E = [m^2 c^4 + (\hbar c/a)^2]^{1/2} \quad (\text{for } k = \pi/a \pm n2\pi/a) \quad (11)$$

in agreement with the curve sketched by Heisenberg. As he notes, this contrasts with the usual ("continuous") case where the energy just grows indefinitely with the wave number  $k$ .

## B. Electrons and protons

Let us first consider Heisenberg's statement that, near the minimum, "the electron behaves quite normally." This can be seen by expanding the cosine in Eq. (8). Taking only the two first terms into account, i.e.,  $\cos ka = 1 - (ka)^2/2$ , gives for the energy near  $k=0$  the expression

$$E = [m^2 c^4 + (\hbar k c)^2]^{1/2}. \quad (12)$$

With the momentum  $p = \hbar k$  it thus coincides with the ordinary relativistic expression. Let us next consider a superposition of plane waves with wave vectors concentrated in a small region around  $k$ , and introduce the group velocity of the wave packet,

$$v_g = d\omega(k)/dk = \hbar^{-1} \partial E(k)/\partial k. \quad (13)$$

With the expression (12) we obtain, again near  $k=0$ , the result

$$v_g \approx \hbar k/m \quad (14)$$

in agreement with the ordinary case.

The behavior near the maxima—with the electron turning into a proton—is more remarkable and not so easily understood. However, we can understand Heisenberg's remarks by making use of the concept of the effective mass. We first recall that according to Ehrenfest's theorem the average value of the acceleration of a quantum particle of mass  $\mu$  subjected to a constant external force  $X$  obeys the relation

$$\langle d^2x/dt^2 \rangle = X/\mu \quad (15)$$

in analogy with Newton's law of motion. From solid state physics, when considering the electrons filling the Fermi band, it is well known that the electron characterized by a wave packet centered around  $k$  behaves like a quasiparticle of effective mass  $\mu^*$  given by

$$1/\mu^* = \hbar^{-2} \partial^2 E(k)/\partial k^2. \quad (16)$$

The average acceleration of an electron moving under the action of a constant electrical field  $F$  directed along the  $x$  axis is therefore  $\langle d^2x/dt^2 \rangle = -eF/\mu^*$ . Insertion of  $E$  from Eq. (8) gives

$$1/\mu^* = c^2 \cos(ka) \Omega^{-1/2} - (\hbar c^2/a)^2 \sin^2(ka) \Omega^{-3/2} \quad (17)$$

with the abbreviation

$$\Omega(k) = [m^2 c^4 + (2\hbar^2 c^2/a^2)(1 - \cos ka)].$$

For  $k = \pi/a$  the electron mass term  $mc^2$  can be disregarded since, with Heisenberg's choice of  $a$ ,  $\hbar c/a \gg mc^2$ . The result is a negative effective mass:

$$\mu^* = -2\hbar/ac. \quad (18)$$

The equation of motion of an electron for which  $k \sim \pi/a$  is thus

$$\langle d^2x/dt^2 \rangle = (eac/2\hbar)F, \quad (19)$$

i.e., the electron reacts like a particle (on an empirical point of view, is a particle) with a positive charge  $e$  and a mass equal to  $2\hbar/ac \sim M$ . According to Heisenberg, the ordinary electron is described by the behavior of the wave function

around  $k=0$  and as  $k$  increases (modulo  $2\pi/a$ ) it becomes a proton at rest. Evidently, Heisenberg interpreted the negative sign of the effective mass as a positive electrical charge and not as a negative mass.

As Heisenberg remarks, the self-energy of the electron will be of the order of magnitude  $e^2/a$ , the classical expression for an extended electron. With  $a = \hbar/Mc$  this corresponds to a mass (self-mass) of  $Me^2/\hbar c$  so that the electron-to-proton mass ratio should be of about the same order of magnitude as the reduced fine structure constant. The order-of-magnitude equality between the dimensionless quantities  $M/m$  (ca. 1840) and  $\hbar c/e^2$  (ca. 860) was often seen as significant at the time. As mentioned, the same agreement was pointed out by Ambarzumian and Iwanenko in their sketch of a lattice world from the summer of 1930.

We now consider Heisenberg's remarks concerning the charge density and its variation with the velocity. With the charge density proposed by Heisenberg,

$$\rho = -e(u_n^* u_{n+1} + u_{n+1}^* u_n) \quad (20)$$

the charge will change sign around  $k = \pi/a$ . This is evident from the fact that  $u_n = -u_{n+1}$  exactly at  $k = \pi/a$ , which follows from  $u_{n+1}(x) = u_n(x+a)$  and expression (5). So for this value of the wave number the electron acts as a positively charged source. In the 1930s the standard way of expressing the coupling between, e.g., a Coulomb field and a charged particle was through a term of the Lagrangian density of the form

$$q\phi(r)\psi(r)\psi^*(r),$$

where  $q$  is the charge of the particle,  $\phi(r)$  is the Coulomb field operator, and  $\psi(r)$  is the wave function of the particle. Since  $\psi(r)\psi^*(r)$  is the probability density of the particle, the charge density is just the product of the charge and the probability density, that is,  $q\psi(r)\psi^*(r)$ . We can transcribe Heisenberg's proposal for the linear charge density to

$$\rho = -e[\psi^*(x)\psi(x+a) + \psi^*(x+a)\psi(x)]. \quad (21)$$

For  $a \rightarrow 0$ , where the ordinary description is regained, we get

$$\rho = -2e\psi(x)\psi^*(x).$$

For reasons of correspondence with the continuous case this means that Heisenberg's expression (2) has to be divided by a factor 2. The wave function entering in the charge density also has to be normalized. Let us, as an example, consider a finite part of the  $x$  axis from  $-L$  to  $+L$  and use the wave function

$$u(x) = (2L)^{-1/2} \exp(ikx).$$

Then, by inserting into Eq. (20) and taking the factor 1/2 into account, we get

$$\rho = (-e/2L) \cos ka. \quad (22)$$

We notice that the distribution of charge does not depend on the value of  $x$ . Since a plane wave has no preferred localization in space, this is what we would expect from the ordinary case.

According to Heisenberg, the electrical charge is not constant in the discrete case. To see this, we expand  $\rho$  from Eq. (22) around  $k=0$ :

$$\rho = -e(1 - k^2 a^2/2 + \dots)/2L. \quad (23)$$

We saw previously [Eq. (14)] that to the same order the velocity is given by  $\hbar k/m$ , which implies that the charge of

the electron is a function of the velocity. The absolute value of the electronic charge is  $e$  for  $v=0$  and decreases with velocity. With  $E=h\nu$ , where  $\nu$  is the frequency, Eq. (12) can be written

$$(\hbar k)^2 = (h\nu/c)^2 - m^2 c^2.$$

If  $k$  from this relationship is inserted into Eq. (23), we obtain the charge density as a function of the frequency:

$$\rho(\nu) = -e(\pi^2/2L)\{1 - (h\nu/Mc^2)^2 + (m/M)^2 + \dots\}. \quad (24)$$

This justifies Heisenberg's remark that the (relative) deviation from the ordinary case—where  $\rho$  is independent of  $\nu$ —is of the order  $(h\nu/Mc^2)^2$ .

A charge density of the form (20) or (21) indicates, if only implicitly, the germinal idea behind the class of so-called nonlocal field theories. The basic idea of such theories is (or was) that elementary particles cannot be exactly localized in some space point at a given instant of time. By considering the particle as being "smeared out" over a finite region of the space time continuum, some theoreticians of the 1930s—including Gleb Wataghin, Fred Hoyle, and Moisei Markov—attempted to build up nondivergent field theories.<sup>11</sup> The nonlocal field approach, which attracted considerable attention in the early 1950s, is evidently related to the ideas of a possible discreteness of space time or, alternatively, to the idea of an intrinsic limitation in the precision of determining space and time coordinates of a particle. In Heisenberg's one-dimensional example we see the nonlocal field philosophy *in embryo*. The expression of the charge density, or the corresponding Lagrangian, shows that the particle-field interaction at some point  $x$  is not only determined by  $\psi(x)$ , but also by  $\psi(x+a)$ . This is, essentially, the nonlocal field idea, and as such it will not be too far-fetched to announce Heisenberg's unpublished theory as perhaps the first (and possibly unconscious) germ of the idea.

### C. Heavy photons

In order to understand Heisenberg's remarks about light quanta (photons) we consider again Eq. (8), this time with  $m=0$ . In the discrete world, the energy of photons is given by

$$E_\gamma = (\hbar c/a)[2(1 - \cos ka)]^{1/2} \quad (25)$$

and for  $k \sim 0$  we have the usual result  $E_\gamma = \hbar kc = pc$ , or  $\omega = kc$ . The energy curve is sketched in Fig. 2.

Another surprising result arises for photons when  $k \sim \pi/a$ . If we apply expression (13) for the group velocity using the dispersion law as given by Eq. (25), we get

$$v_g = c \sin(ka)[2(1 - \cos ka)]^{-1/2}. \quad (26)$$

For  $k \sim 0$  the result is  $v_g \approx c$ , but when  $k = \pi/a$ , or  $\lambda = 2a \sim h/Mc$ , we have that  $v_g \approx 0$ , in strong contrast with the behavior of ordinary photons.

A series expansion around  $k=0$  further illuminates Heisenberg's remarks. From Eq. (26) we have

$$v_g = c[1 - k^2 a^2/3! + \dots].$$

With  $k = 2\pi\nu/c$  and  $a = h/Mc$  the expression becomes

$$v_g = c\{1 - (4\pi^2/3!)(h\nu/Mc^2)^2 + \dots\}. \quad (27)$$

We see that "the velocity of light must thus be a function of the frequency, the (relative) deviations from  $c$  being of the order  $(h\nu/Mc^2)^2$ ," as Heisenberg claims. For small values of

$k$  light behaves in the usual way, but as  $k$  increases (modulo  $2\pi/a$ ) the velocity of light becomes slower and slower. At  $k = \pi/a$  the velocity is exactly zero. In this case we get for the (rest) energy

$$E_\gamma = 2\hbar c/a \equiv Mc^2,$$

i.e., the photon behaves as it has gained protonic mass. In this respect the photon only differs insignificantly from the electron, cf. Eq. (18). What distinguishes the solution for photons near  $k = \pi/a$  is that it evidently refers to something electrically neutral. For this reason Heisenberg speaks of "(slow) light quanta of mass  $M$ ." Actually, these "photons" were merely introduced (for  $k=0$ ) as massless solutions to the discrete Klein-Gordon equation, and, apart from being neutral, they have little in common with ordinary photons. In 1930, the photon was the only known neutral elementary particle, with or without mass, and so it was natural for Heisenberg to refer to the hypothetical neutral particles appearing in his lattice world as light quanta. He might also have called them neutrons, but this name was already used by Rutherford and others for hypothetical composites of protons and electrons. We shall return to the possible connection between heavy quanta and neutrons in Sec. VII.

### D. Nonconservation in one dimension

According to Heisenberg, neither momentum, energy, nor charge is conserved in the lattice world where Lorentz invariance is also violated. In fact these unpleasant features are not peculiar to the three-dimensional lattice world but are also properties of the one-dimensional case considered by Heisenberg. Let us point out that his claim for nonconservation of momentum and energy is to be understood only within the circumstances we shall discuss.

The lack of ("decent") Lorentz invariance is obvious from the fact that only space is discretized, whereas time is not. As a minimum requirement for a relativistic theory, space and time must be treated on the same footing. Also, as pointed out by both Bohr and Ehrenfest (see Sec. VI), Heisenberg's fundamental length  $a$  is not relativistically invariant; as all lengths, it is subject to the Fitzgerald-Lorentz contraction. We have already seen that the electrical charge varies with the velocity and so cannot be a conserved quantity.

Nonconservation of momentum—really of quasimomentum—follows from the close analogy with Peierls' theory of metal lattices in which the quasimomentum is not conserved (see Sec. VI). Whereas momentum nonconservation follows nicely from the solid state analogy, energy nonconservation does not. Heisenberg's statement can only be understood if we assume that he considered to also replace the time derivative by a finite difference. In that case the separated time part of the Klein-Gordon equation,

$$-\hbar^2 \partial^2 f(t)/\partial t^2 = E^2 f(t), \quad (28)$$

must be replaced by

$$-\hbar^2 [f(t+\tau) - 2f(t) + f(t-\tau)]/\tau^2 = E^2 f(t), \quad (29)$$

where  $\tau$  is a smallest time interval. By proceeding in complete analogy with the space case, i.e., writing

$$f_n(t) = f(\Theta + n\tau) = \exp[i\omega(\Theta + n\tau)],$$

the energy eigenvalues

$$E = \pm (\hbar/\tau)[2(1 - \cos \omega\tau)]^{1/2} \quad (30)$$

are found. The energy is thus continuous and periodic in  $\omega$ , and only in the limit  $\omega \rightarrow 0$  will its absolute value be the familiar  $\hbar\omega$ . This quantity,  $\hbar\omega$ , corresponds to the quasimomentum  $\hbar k$  and is thus a quasienergy. Since  $\omega$  is a multivalued function of  $E$ , even if the energy is exactly known the temporal behavior of the wave function is still undetermined. A general solution for the time dependence of a state with a definite energy corresponding to the frequency  $\omega$  can be written

$$f(t) = g(t) \exp(i\omega t),$$

where  $g(t) = g(t+T)$  is periodic in time with the period  $T$ . Now consider a time-dependent sinusoidal perturbation of frequency  $\omega_p$  so that the interaction term is proportional to  $\exp(i\omega_p t)$ . For simplicity, let us refer to the first order time-dependent perturbation theory. According to it, when the time dependence of the initial and final states are given by the monochromatic waves  $\psi_i = \exp(i\omega_i t)$  and  $\psi_f = \exp(i\omega_f t)$ , the transition probability from state  $i$  to  $f$  is zero unless  $\omega_f = \omega_i + \omega_p$ . But if we consider the general solutions  $g(t) \exp(i\omega t)$  instead of pure waves, then transitions for which

$$\omega_f = \omega_i + \omega_p + \eta, \quad \text{where } \eta = n2\pi/T$$

with  $n = 0, \pm 1, \pm 2, \dots$  (31)

are also possible. Our considerations are analogous to Peierls' discussion of nonconservation of quasimomentum, and we assume that Heisenberg just followed the analogy when speaking of energy nonconservation. We further observe that not only is the quasienergy  $\hbar\omega$  not conserved, but neither is the energy  $E$ . For example, if  $\omega_p$  has one of the  $\eta$  values, a state of energy  $E$  may absorb an energy quantum  $\hbar\omega_p$  and still be in the same energy state as before.

### E. The three-dimensional lattice world

The energy eigenvalues of an electron moving in a lattice world of three dimensions are obtained by a straightforward generalization of the approach in Sec. IV A. This consists of introducing a discrete Laplacian:

$$\Delta_x^2 + \Delta_y^2 + \Delta_z^2,$$

where  $\Delta_x^2$  operates only on the  $x$  variable, etc. This finite difference operator defines a preferred reference frame and can be expressed only in Cartesian coordinates. Let us assume the eigenfunction  $U(x, y, z)$  to be factorized as  $u_x(x)u_y(y)u_z(z)$ , where  $u_x = w_1(x) \exp(ik_x x)$  and  $w_1(x) = w_1(x+a)$ , and similarly for  $y$  and  $z$ . Then the energy eigenvalues of the stationary Klein-Gordon equation become

$$E = \pm [m^2 c^4 + 2(c\hbar/a)^2 \{(1 - \cos k_x a) + (1 - \cos k_y a) + (1 - \cos k_z a)\}]^{1/2}. \quad (32)$$

The energy has a maximum absolute value, and the astonishing results from before still hold when each of the wave vector components  $k_x, k_y, k_z$  assume the particular values encountered in the one-dimensional case. It may be pointed out that the introduction of a discrete Laplacian implies lack of space isotropy because it is not invariant under continuous rotation. The anisotropy is the only feature which is peculiar to the three-dimensional case.

## V. IMPACT OF SOLID STATE THEORY

In 1930, Heisenberg was occupied with the problems of quantum electrodynamics and was just beginning to think about the mysterious atomic nucleus and the (almost equally mysterious) cosmic radiation. But in addition to these branches of frontier theoretical physics he also cultivated an interest in the application of quantum mechanics to solid state physics. Under Heisenberg's leadership, the institute in Leipzig quickly became a center in this area which was pioneered in particular by two of Heisenberg's first students, Felix Bloch and Rudolf Peierls.<sup>12</sup> Heisenberg's expert knowledge of the new quantum mechanical theory of metals served as an inspiration for his solution of the lattice world problem.

The foundation of the quantum theory of electrons in lattices was laid by Bloch in his thesis of 1928, but it was the subsequent work of Peierls that proved most important to Heisenberg in his speculations about a lattice world. In a paper of 1929, Peierls investigated, on Heisenberg's suggestion, the so-called anomalous or "positive" Hall effect in which a current in some metals is influenced by the magnetic field as if the conduction carriers were positively charged.<sup>13</sup> He studied the energy function  $E(k)$ , where  $\hbar k$  is the crystal momentum, and found that for electrons in the upper part of the band  $E(k)$  will be negatively curved. As he realized, this corresponds to a group velocity that decreases with the momentum, contrary to the behavior of free electrons. Peierls later recalled his work as follows:<sup>14</sup>

[I] first had to convince myself ... that the mean velocity of the electron was given by  $dE/dk$ , and therefore different from that for a free electron of the same  $k$ , if the energy function  $E(k)$  was different. It was obvious, in particular, that in Bloch's tight-binding model the energy would flatten off near the band edge, so that the current would there go to zero. Thus, for an electron near the band edge an electric field could cause a decrease, rather than an increase, in the velocity in the field direction.

In Peierls' work the concepts of effective mass and holes were not introduced explicitly, but they were there implicitly. In the early part of 1930 specialists in the area recognized that unfilled states near the top of an otherwise filled band behaved as they were positively charged electrons (with positive mass). The idea of an effective mass first appeared in print (under the name "apparent mass") in a paper by Léon Brillouin of the summer of 1930,<sup>15</sup> and one year later Heisenberg explicitly introduced the concept of solid state holes by means of quantum mechanics. He concluded that "the holes move exactly like electrons with positive charge under the influence of a disturbing external field."<sup>16</sup>

Also Peierls' theory of metal lattices, worked out after he had become Pauli's assistant in Zurich, inspired Heisenberg. In this work,<sup>17</sup> Peierls considered the transition of an electron from one energy state to another with the simultaneous emission or absorption of a phonon. He showed that in general the total "momentum"  $\Sigma \hbar k$ , i.e., the sum of the phonon's momentum and the electron's quasimomentum—will differ from the total final momentum by a reciprocal lattice vector (times  $\hbar$ ). This effect, a result of the periodic factor in the general form of the energy eigenfunctions, is associated with the so-called Umklapp process introduced by Peierls in 1929.

As is evident from his letter to Bohr, and from our technical interpretation of it, Heisenberg considered his lattice world in analogy with the very recent knowledge obtained about the behavior of electrons in metals. In particular, he

used Peierls' theory as a tool in explaining how electrons could metamorphose into protons for certain values of the wave number. Insofar as Heisenberg's idea of a lattice world belongs to elementary particle physics, this was the first time that methods from solid state physics were transferred to the apparently very different field of fundamental particles. It is no accident that this first case of theory transfer took place in the Leipzig institute, where the conditions were ideal for cross-disciplinary work. As Bloch noticed many years later: "There was so much interplay between all the physicists ... that as soon as somebody had an idea, another one took it up and put it in a different form and used it somewhere else."<sup>18</sup>

The interplay also manifested itself in quantum electrodynamics being applied to solid state theory at an early stage. Thus, in 1930 Igor Tamm applied the new Heisenberg–Pauli theory in a study of scattering of light in crystals. It was in this work that quanta of elastic oscillations—phonons—were first introduced.<sup>19</sup> (The name "phonon" was only coined two years later, by Yakov Frenkel.)

## VI. RESPONSES TO THE LATTICE WORLD

Heisenberg's theory of a lattice world was as radical as it was ambitious. For a brief period he believed to be on his way to construct a unitary theory of electrons and protons based on his notion of a fundamental length, a theory which aspired to the same high goals that motivated Dirac in his contemporary theory of holes. It is remarkable that Heisenberg was willing to seriously entertain a theory that deviated so drastically from established physics with regard to fundamental principles such as space isotropy, relativistic invariance, and conservation of energy, momentum, and charge.

When Heisenberg wrote to Copenhagen about his ideas, Bohr was well acquainted with the general framework of Heisenberg's idea, the hypothesis of a smallest length. In October 1929, Nevill Mott offered him "a proof that you cannot measure the position of a particle to more than a certain degree of accuracy," which Mott found to be  $h/mc$ .<sup>20</sup> However, Bohr judged Mott's argument to be invalid if relativity was taken into account. With regard to the idea of an absolute uncertainty in position, as proposed by Flint and Richardson, and now also by Mott, Bohr wrote: "To my view all such limitations would interfere with the beauty and consistency of the theory [of relativity] to far great an extent. The only limitations in the relativistic theory which I think possible are those connected with the problem of the constitution of the electron."<sup>21</sup> One week before the beginning of the 1930 Easter conference, Bohr returned to the subject, now with a very different view:<sup>22</sup>

I have in these days been thinking intensely over the whole problem of the limits of observation. You remember that I could not agree in your arguments for an absolute limit as regards space determination of electrons as that proposed by Richardson. Although I still think that my arguments were correct, I have revised my attitude towards the matter and think now that very general arguments can be given in favour of such a limitation of space determinations, and that this very point is of essential importance as regards obtaining a consistency in the apparent chaos of relativity quantum mechanics.

The change in Bohr's attitude may have been influenced by his correspondence with Heisenberg. It shows, at any rate, that Bohr was fully prepared to accept the idea of an absolute uncertainty in position (although not, necessarily, the related idea of a smallest length).

Although in general sympathy with Heisenberg's approach, Bohr did not share his enthusiasm. After having discussed the letter of March 10 with Oskar Klein, he decided that the lattice world was not the answer to the troubles of quantum theory. In a letter to Heisenberg, Bohr admitted "the beauty in the idea of connecting so inviolably the existence of the electron and proton," but objected that Heisenberg's view was difficult to reconcile with correspondence considerations. The possible violation of energy and momentum conservation in Heisenberg's lattice world was not a shocking feature to Bohr, who himself argued for such violation in the case of beta decay,<sup>23</sup> but he considered nonconservation of electrical charge to be much more problematical. Pauli, who was opposed to all such ideas, wrote to Klein in Copenhagen in March 1930. Referring to the speculations of energy nonconservation, he stated that Bohr had recently found "a powerful ally" in Heisenberg—probably a reference the lattice world theory.<sup>24</sup>

Bohr saw Heisenberg's new idea as belonging to the same category as Dirac's hypothesis of holes and believed that it could be criticized in the same way. According to Bohr, "conceptions of charge and radiation must be founded directly on classical electrodynamics," and this made him doubt the validity of Heisenberg's smallest length. "I am not even sure," he wrote, "that the quantity  $h/Mc$  can always be considered the absolute limit of the application of spatial dimensions in connection with classical theories. Although this quantity, according to classical electrodynamics, is just the electron's "self-radius," then the very relativity contraction seems to indicate the possibility that, under some circumstances, one can attach a well-defined meaning to a more exact determination of the position at least in a single direction."<sup>25</sup> To make his position clear, Bohr enclosed in his letter to Heisenberg a copy of an unpublished manuscript from the spring of 1929 in which he argued for violation of energy conservation in beta decay processes.<sup>26</sup> He also enclosed a copy of a letter to Dirac, an early reply to Dirac's idea of protons as holes in a "sea" of negative-energy electrons. In this letter, Bohr made it clear that his willingness to abandon energy conservation did not extend to charge conservation. As he wrote to Dirac (and Heisenberg):<sup>27</sup>

In the fact that the total charge of the nucleus can be measured before and after the  $\beta$ -ray disintegration and that the results are in conformity with conservation of electricity I see a support for upholding the conservation of the elementary charges even at the risk of abandoning the conservation of energy, and I do not quite understand your reasons for taking the opposite view. Of course, I do not wish to advocate any of the scepticism of old and new as to the strict conservation of energy in ordinary quantum theory.

A few days later, Heisenberg acknowledged Bohr's objections and admitted that the lattice world idea was probably not tenable: "I understand quite well your objections against my attempt with the finite space cells, and I agree completely with you that this line of attack is much too crude."<sup>28</sup>

During the Easter conference at Bohr's institute, held April 8–15, the lattice idea was discussed among the participants who included Heisenberg, Peierls, Bloch, Landau, Gamow, Møller, Rice, and others. Tatiana Ehrenfest, who was also at the conference, described to her husband, Paul Ehrenfest, what the situation in quantum physics looked like from Copenhagen:<sup>29</sup>

Tanitschka was in Copenhagen for the Bohr conference

and has sent me a comprehensive and good account of it.... All attempts of quantizing the electromagnetic fields have got stuck in the mud (infinite mass of the point electron, Dirac difficulties). It seems that Dirac's "hole physics" has not at all Bohr's sympathy (but of course he is, as usual, a polite man). Heisenberg cut capers with a "granulated ether" (the grain size of which is of course not relativistic invariant), but, if possible (!), he believes in the madness even less than the others.

At that time, when the lattice world was only about one month old, Heisenberg seems to have stopped regarding it as a possible theory of the real world and merely to have discussed it as an interesting academic question.

The possibility of solving the troubles of quantum theory by introducing a discrete space time was among the topics discussed at the All-Union Physical Congress held in Odessa in August 1930. Inspired by the attempts of Heisenberg and of Iwanenko and Ambarzumian, the young Soviet physicist Matvei Bronstein took up the idea of a lattice world. He discussed "quantization of space" with other physicists at the Odessa congress (including Pauli, Peierls, Tamm, and Frenkel) and tried to circumvent some of the fundamental problems that had forced Heisenberg to abandon the lattice world approach.<sup>30</sup> However, Bronstein's efforts to breathe new life into the idea failed. Pauli remained unconvinced and declared to Bronstein that "Those who are making holes in continuous space should mind where they step."<sup>31</sup>

## VII. THE LATTICE WORLD AFTER 1930

In spite of its brief life, Heisenberg's theory sketch of March 1930 was not without implications. For one thing, the failure forced Heisenberg to change his research program and admit that, for the time being, the study of high-energy particles was more likely to secure progress in quantum electrodynamics. This led him to focus on cosmic rays and, until 1932, to abandon the atomic nucleus as a hopeless case.<sup>32</sup>

A brief comment on Heisenberg's lattice world as an early nuclear model may be appropriate at this time. We have not been able to justify his claim that "in order to build up wave packets of nuclear dimensions, only waves close to the maximum of the  $E$  curves can be used." The reasons are the following. Heisenberg must have considered a three-dimensional packet, and let us assume that it is expressed as the product of three Gaussians,  $G(x)G(y)G(z)$ . We can give it whatever dimensions we like, and, the Fourier transform still being a Gaussian, the density of component waves in each wave vector component space is maximum near zero. Therefore, Heisenberg's statement can be true only for particular cases, not in general.<sup>33</sup>

All the same, from a historical point of view we must just accept Heisenberg's claim, which leads to an electron-free nucleus. About 1930 it was realized that somehow electrons "ought" not to be in the nucleus and that their presence caused all sorts of problems. However, in spite of much speculation no viable alternative to the proton-electron model emerged until the neutron was discovered.<sup>34</sup> In Heisenberg's remarkable picture of the nucleus there are no electrons, which are replaced by the strange heavy photons playing a role not unlike the later neutrons. The expulsion of nuclear electrons was an advantage, but of course the alternative was highly hypothetical and posed many new questions (such as the origin of beta electrons).

Interestingly, at the same time as Heisenberg developed his lattice world idea, Ambarzumian and Iwanenko at-

tempted to apply Dirac's hole theory to explain beta processes. They suggested that "within the nucleus, the electrons lose, in a certain sense, their individuality ... in the same way as photons [when absorbed in an atom]."<sup>35</sup> Also George Gamow, then in Cambridge, tried to make use of Dirac's theory in his attempts to understand the nucleus. In a long letter to Bohr of February 25, he admitted his failure and concluded that the atomic nucleus was as frustrating a problem as ever. "This is all rather bad," he wrote, "and Dirac is also very sorry about it."<sup>36</sup>

It is tempting to relate Heisenberg's 1930 lattice world to his famous nuclear theory of 1932 in which the recently discovered neutron was first incorporated. After all, as far as mass and charge are concerned the heavy photons are suggestively like neutrons. However, not only is there no documentary evidence for such a generic relationship, but at first (in 1932) Heisenberg conceived the neutron as a proton-electron composite and not an elementary particle. And yet, as suggested by Joan Bromberg, there may have been a connection.<sup>37</sup> Although Heisenberg's attempt to construct a nuclear model of protons and neutral particles failed in 1930, it can hardly have left his mind completely. The attempt indicates that he was mentally prepared for the neutron as a nuclear constituent when the opportunity came with Chadwick's discovery two years later. One may assume that Heisenberg, as a result of his earlier attempt, was more receptive to the neutron than most physicists. In this connection it may also be relevant to recall that the other pioneer of the proton-neutron model, Iwanenko, had himself suggested a lattice world and was familiar with Heisenberg's idea. As Iwanenko, together with Ambarzumian, had likened nuclear electrons with photons in 1930, so he argued in 1932 that "the electrons in nuclei are really quite analogous to the absorbed photons."<sup>38</sup>

Whatever the influence on nuclear theory, Heisenberg did not forget about his early attempt, and in 1936-38 the smallest length reappeared in a version that to some extent incorporated elements of his idea of 1930. The aim was again to formulate a divergence-free quantum electrodynamics, which he attempted to do in 1936 by using  $\sqrt{f}$ , with  $f = g/\hbar c$  the Fermi constant of weak interactions, as a critical length.<sup>39</sup> After this idea proved untenable, Heisenberg formulated two years later a theory of a smallest length based on the Yukawa meson theory. In the theory of 1938, the smallest length was given by  $\hbar/\mu c$  where  $\mu$  is the mass of the Yukawa meson.<sup>40</sup> Although the high hopes Heisenberg placed on his new theory never materialized, it played an important role in his thinking in the late 1930s and elements of it were incorporated in the  $S$ -matrix theory that Heisenberg developed during the war. The continuity in Heisenberg's thoughts between 1930 and 1938 was centered around the utility of a smallest length.

After 1945, many physicists attempted to complete Heisenberg's program of establishing divergence-free quantum field theories based on lattice space time ideas.<sup>41</sup> Although much of this work was close to what Heisenberg had done in the 1930s, the new generation of physicists seems not to have been familiar with his theories of a smallest length to which they rarely referred. And it was of course ignorant of his 1930 lattice world which contained the germs of many of the postwar theories of discrete space time.

## VIII. CONCLUSIONS

In Heisenberg's lattice world of 1930 we see the confluence of several ideas and approaches. The immediate problem was the electron's infinite self-energy according to quantum electrodynamics which Heisenberg, in accordance with many of his colleagues, conceived as connected with the problems of understanding the atomic nucleus. As it was only glimpsed at the time, these problems were rooted in the shared belief in nuclear electrons (however unwelcome they were), which again related to the difficulty of explaining the continuous beta spectrum. In Heisenberg's attack on this complex of problems he used methods from the new quantum-mechanical theory of metals, applied to a model of space divided into smallest cells. This model was probably inspired by current speculations of a discrete space time and the existence of minimal extensions or uncertainties in space.

It may seem confusing with these different problems and approaches, but the confusion is largely artificial, a result of our natural attempt to separate the problems in clean, distinguishable compartments. However, about 1930 such compartmentalization made little sense, cf. with Bloch's recollection quoted above. To the physicists of the time, including Heisenberg, all the problems and attempts at solution were thoroughly interconnected—so no wonder that the historian of physics is confronted with a situation that appears confusing and difficult to separate in the “elements” the analyst might want to have.

Heisenberg's aborted theory reflected a situation in quantum physics that made many leading physicists believe that drastic departures from ordinary physics were needed in order to overcome the difficulties. The lattice world was only one of the revolutionary approaches of the period, which also witnessed Bohr's idea of energy nonconservation, Dirac's theory of holes, and Pauli's theory of neutrinos. From this point of view, Heisenberg's radical theory was quite representative of the revolutionary attitude favored by many physicists.

Heisenberg did not accept Dirac's identification of protons with holes (i.e., antielectrons), but in some respects his lattice world expressed the same aspirations that characterized Dirac's hole theory. They were both led to a unitary view—“the dream of philosophers” according to Dirac—in which the proton and the electron were not independent particles but manifestations of the same fundamental particle or field. This was a feature that Heisenberg noticed with satisfaction, but it was less important to him than it was for Dirac. The British theoretician was deeply committed to the unitary view which acted as a strong motive for his proton-cum-antielectron hypothesis.<sup>42</sup> Heisenberg, on the other hand, seems merely to have considered it an unexpected extra bonus of a theory that he found attractive primarily because it avoided the infinite self-energy of the electron and promised a better understanding of the atomic nucleus. The different emotional commitments to the unitary view resulted in different responses when the two theories were shown to be untenable: Dirac kept to his theory as long as possible and when he left it (in order to replace it with his hypothesis of positive electrons) he regretted that he was forced to admit the separate existence of electrons and protons as a contingent fact. Heisenberg just shelved his lattice world and went on to other work.

Finally, we call attention to the theory transfer from solid state to particle physics. This kind of transfer, where methods from a less fundamental branch of physics, are used in a

more fundamental one (and vice versa), was later applied by Lev Landau and, in particular, Kenneth Wilson. The important work of Wilson, for which he was awarded the Nobel Prize in 1982, brought renormalization group ideas into solid state physics,<sup>43</sup> and in its further development it led to the first lattice gauge theories.<sup>44</sup> It is interesting that these modern theories, dating back to 1974, have elements in common with Heisenberg's old lattice world, if only in terminology.

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<sup>1</sup>Werner Heisenberg, “Die Selbstenergie des Elektrons,” *Z. Phys.* **65**, 4–13 (1930). Received August 3, 1930.

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<sup>3</sup>W. Heisenberg and Wolfgang Pauli, “Zur Quantendynamik der Wellenfelder,” *Z. Phys.* **56**, 1–61 (1929). Silvan S. Schweber, *QED and the Men Who Made It: Dyson, Feynman, Schwinger, and Tomonaga* (Princeton University, Princeton, NJ, 1994), pp. 39–45.

<sup>4</sup>Gottfried Beck, “Die Zeitliche Quantelung der Bewegung,” *Z. Phys.* **53**, 675–682 (1929). For other suggestions of time atoms, see Helge Kragh and Bruno Carazza, “From time atoms to space-time quantization: The idea of discrete time, ca. 1925–1936,” *Stud. Hist. Philos. Sci.* **25**, 437–462 (1994).

<sup>5</sup>Henry T. Flint, “Relativity and the quantum theory,” *Proc. R. Soc. London Ser. A* **117**, 630–637 (1928); Arthur E. Ruark, “The limits of accuracy in physical measurements,” *Proc. Natl. Acad. Sci.* **14**, 322–328 (1928); Kragh and Carazza, “From time atoms to space-time quantization.”

<sup>6</sup>Victor Ambazumian and Dmitri Iwanenko, “Zur Frage nach Vermeidung der unendlichen Selbststrückwirkung des Elektrons,” *Z. Phys.* **64**, 563–567 (1930), received July 21, 1930. According to Ambazumian and Iwanenko, the idea of a space time lattice world was also suggested (informally) by Pascual Jordan. In 1975, Iwanenko referred briefly to Heisenberg's 1930 lattice world, which he described as an attempt to formulate a theory of the atomic nucleus without electrons. D. Iwanenko, “Die Entstehung des Atomkernmodells,” in *Das Neutron: Eine Artikelsammlung*, edited by Boris M. Kedrow (Akademie-Verlag, Berlin, 1979), pp. 20–40. Russian original 1975, p. 27.

<sup>7</sup>In a letter to Dirac of January 16, 1930, Heisenberg criticized Dirac's theory, but added that “so far, ... I don't know anything better than your theory.” See Helge Kragh, *Dirac: A Scientific Biography* (Cambridge University, Cambridge, 1990), p. 95. We take this as indicating that Heisenberg had not yet started contemplating his idea of a lattice world.

<sup>8</sup>Heisenberg to Bohr, February 26, 1930 (Bohr Scientific Correspondence, BSC, 20.2).

<sup>9</sup>Heisenberg to Bohr, March 10, 1930 (BSC, 20.2).

<sup>10</sup>Felix Bloch, “Über die Quantenmechanik der Elektronen in Kristallgittern,” *Z. Phys.* **52**, 555–600 (1928).

<sup>11</sup>For a very preliminary historical survey of his class of theories, see B. Carazza and H. Kragh, “Some attempts of time discretization in modern physics,” in *History of Physics in Europe in the 19th and 20th Centuries*, edited by Fabio Bevilacqua (Italian Society of Physics, Bologna, 1993), pp. 269–275. See also W. A. McKinley, “Search for a fundamental length in microscopic physics,” *Am. J. Phys.* **28**, 129–134 (1960).

- <sup>12</sup>For details, see Nevill Mott, ed., "The beginnings of solid state physics," Proc. R. Soc. London Ser. A **371**, 1–177 (1980); Lillian Hoddeson, Gordon Baym, and Michael Eckert, "The development of the quantum-mechanical theory of metals, 1928–1933," Rev. Mod. Phys. **59**, 287–327 (1987); *Out of the Crystal Maze: Chapters from the History of Solid State Physics*, edited by L. Hoddeson et al. (Oxford University, Oxford, 1992), pp. 88–181.
- <sup>13</sup>Rudolf E. Peierls, "Zur Theorie der galvanomagnetischen Effekte," Z. Phys. **53**, 255–266 (1929).
- <sup>14</sup>R. E. Peierls, "Recollections of early solid state physics," pp. 28–38 in Mott ed. "Beginnings of solid state physics," see Ref. 12, p. 30.
- <sup>15</sup>Léon Brillouin, "Les électrons libres dans les métaux et le rôle des réflexions de Bragg," J. Phys. (Paris) **1**, 377–400 (1930).
- <sup>16</sup>W. Heisenberg, "Zum Paulischen Ausschlussprinzip," Ann. Phys. **10**, 888–904 (1931). The holes dealt with by Heisenberg should not be confused with the holes in the negative-energy states that were suggested by Dirac at the same time. There appears to have been no relationship in the development of the two concepts except that they both relied on Pauli's exclusion principle. See Hoddeson, Baym, and Eckert, "The development of the quantum-mechanical theory of metals," pp. 294–295 (see Ref. 12).
- <sup>17</sup>R. E. Peierls, "Zur kinetischen theorie der Wärmeleitung in Kristallen," Ann. Phys. **3**, 1055–1101 (1929), received October 24, 1929; "Zur theorie der elektrischen und thermischen Leitfähigkeit von Metallen," Ann. Phys. **4**, 121–148 (1930), received December 6, 1929.
- <sup>18</sup>Interview with F. Bloch, conducted by L. Hoddeson December 15, 1981, as quoted in Hoddeson et al., eds. Ref. 12, p. 114.
- <sup>19</sup>I. Tamm, "Über die Quantentheorie der molekularen Lichtzerstreuung in festen Körpern," Z. Phys. **60**, 345–363 (1930).
- <sup>20</sup>Mott to Bohr, October 1929 (BSC, 14.4). Mott was inspired by "the funny paper of (Reinhold) Fürth's," presumably a reference to R. Fürth, "Über einen Zusammenhang zwischen quantenmechanischer Unschärfe und Struktur der elementarteilchen und eine hierauf begründete Berechnung der Massen von Proton und Elektron," Z. Phys. **57**, 439–446 (1929).
- <sup>21</sup>Bohr to Mott, October 18, 1929 (BSC, 14.4).
- <sup>22</sup>Bohr to Mott, April 1, 1930 (BSC, 23.4). Bohr's reference to Owen Richardson was to H. T. Flint and O. W. Richardson, "On a minimum proper time and its applications (1) to the number of the chemical elements (2) to some uncertainty relations," Proc. R. Soc. London Ser. A **117**, 637–649 (1928).
- <sup>23</sup>Niels Bohr, "Chemistry and the quantum theory of atomic constitution," J. Chem. Soc. **1932**, 349–384. This article was a revised version of the Faraday Lecture which Bohr gave on May 8, 1930.
- <sup>24</sup>Pauli to Klein, March 10, 1930, as reproduced in *Wolfgang Pauli: Wissenschaftlicher Briefwechsel*, edited by K. v. Meyenn (Springer, Berlin, 1985), Vol. 2, p. 7. The discussion between Bohr and Pauli concerning violation of conservation principles is analyzed in Carsten Jensen (unpublished dissertation, University of Copenhagen, 1990), Chap. 6.
- <sup>25</sup>Bohr to Heisenberg, March 18, 1930 (BSC, 20.2).
- <sup>26</sup>N. Bohr, "β-ray spectra and energy conservation," in *Niels Bohr, Collected Works*, edited by R. Peierls (North Holland, Amsterdam, 1986), Vol. 9, pp. 85–89.
- <sup>27</sup>Bohr to Dirac, December 5, 1929, as reproduced in Donald F. Moyer, "Evaluation of Dirac's electron," Am. J. Phys. **49**, 1055–1062 (1981).
- <sup>28</sup>Heisenberg to Bohr, March 23, 1930 (BSC, 20.2).
- <sup>29</sup>P. Ehrenfest to G. Dieke, S. A. Goudsmit, and G. Uhlenbeck, April 24, 1930 (AHQP, 64.3).
- <sup>30</sup>Gennady E. Gorelik and Victor Ya. Frenkel, *Matvei Petrovich Bronstein and Soviet Theoretical Physics in the Thirties* (Birkhäuser, Basel, 1994), pp. 39–43. Bronstein described the lattice world approach in an article in a popular science journal: M. P. Bronstein, "Novy krizis teorii kvant," Nauchnoe Slovo **1**, 38–55 (1931).
- <sup>31</sup>Gorelik and Frenkel, *Bronstein and Soviet Theoretical Physics*, Ref. 30, p. 38. Pauli's remark may also have related to Dirac's hole theory, which he disliked.
- <sup>32</sup>Bromberg, "The impact of the neutron, Ref. 2.
- <sup>33</sup>This conclusion is based on our interpretation that the wave functions are defined everywhere in space, the space being continuous and only the space differential operators being replaced by finite difference operators. If one adopts the more radical interpretation, where the wave function is defined only at the lattice points, Heisenberg's statement does not hold any better.
- <sup>34</sup>The perplexing problem of nuclear electrons was discussed in, e.g., Fritz G. Houtermanns, "Neuere Arbeiten über Quantentheorie des Atomkerns," Ergebn. Exakten Naturwiss. **9**, 123–221 (1930); G. Gamow, *Constitution of Atomic Nuclei and Radioactivity* (Clarendon, Oxford, 1931). For historical perspective, see Roger H. Stuewer, "The nuclear electron hypothesis," in *Otto Hahn and the Rise of Nuclear Physics*, edited by William R. Shea (Reidel, Dordrecht, 1983), pp. 19–68.
- <sup>35</sup>V. Ambarzumian and D. Iwanenko, "Les électrons inobservables et les rayons β," C. R. Acad. Sci. (Paris) **190**, 582–584 (1930). Presented March 3, 1930.
- <sup>36</sup>Gamow to Bohr, February 25, 1930 (BSC, 19.4).
- <sup>37</sup>Bromberg, "The impact of the neutron," (see Ref. 2). For Heisenberg's proton-neutron (or, rather, proton-neutron-electron) model, see also Laurie M. Brown and Helmut Rechenberg, "Nuclear structure and beta decay (1932–1933)," Am. J. Phys. **56**, 982–988 (1988).
- <sup>38</sup>D. Iwanenko, "Sur la constitution des noyaux atomiques," C. R. Acad. Sci. (Paris) **195**, 439–441 (1932). See also Stuewer, "The nuclear electron hypothesis," pp. 46–49.
- <sup>39</sup>W. Heisenberg, "Zur Theorie der 'Schauer' in der Höhenstrahlung," Z. Phys. **101**, 533–540 (1936).
- <sup>40</sup>W. Heisenberg, "Über die in der Theorie der Elementarteilchen auftretende universelle Länge," Ann. Phys. **32**, 20–33 (1938). Heisenberg's continued occupation with a smallest length is examined in H. Kragh, "Arthur March, Werner Heisenberg, and the search for a smallest length," Rev. d'Hist. Sci. (to be published).
- <sup>41</sup>For example, Hartland S. Snyder, "Quantized space-time," Phys. Rev. **71**, 38–41 (1947); E. L. Hill, "Relativistic theory of discrete momentum space and discrete space-time," *ibid.*, **100**, 1780–1783 (1955); Henning Harmuth, "Die Unschärferelationen in der Dirac-Gleichungen und in der relativistischen Schrödinger-Gleichung," Z. Naturforsch. Teil A **11**, 101–118 (1956); A. Das, "Cellular space-time and quantum field theory," Nuovo Cimento **18**, 482–505 (1960).
- <sup>42</sup>Paul A. M. Dirac, "A theory of electrons and protons," Proc. R. Soc. London Ser. A **126**, 360–365 (1930); Kragh, *Dirac*, pp. 89–103, see Ref. 7.
- <sup>43</sup>Kenneth G. Wilson, "Renormalization group and critical phenomena," Phys. Rev. B **4**, 3174–3183, 3185–3205 (1971). For the impact of Wilson's work, see A. Pickering, *Constructing Quarks: A Sociological History of Particle Physics* (Edinburgh University, Edinburgh, 1984), pp. 209–13.
- <sup>44</sup>K. G. Wilson, "Confinement of quarks," Phys. Rev. D **10**, 2445–2459 (1974).

### NEWTON'S FIRST LAW

Isaac Asimov, the late biochemist, popularizer of science, and science fiction author, told me that he once had an argument with a theoretical physicist who denied that a dog could know Newton's laws of motion. Isaac asked indignantly, "You say that, even after watching a dog catch a Frisbee with its mouth?"

Murray Gell-Mann, *The Quark and the Jaguar: Adventures in the Simple and the Complex* (W. H. Freeman and Company, New York, 1994), p. 19.